A contribution to the second neighborhood problem

Salman GHAZAL[1]

Abstract

Seymour's Second Neighborhood Conjecture asserts that every digraph (without digons) has a vertex whose first out-neighborhood is at most as large as its second out-neighborhood. It is proved for tournaments, tournaments missing a matching and tournaments missing a generalized star. We prove this conjecture for classes of digraphs whose missing graph is a comb, a complete graph minus 2 independent edges, or a complete graph minus the edges of a cycle of length 5.

1 Introduction

In this paper, graphs are finite and simple. Directed graphs (digraphs) are orientations of graphs, so they do not contain loops, parallel arcs, or digons (directed cycles of length 2). Let $G = (V, E)$ be a graph. The neighborhood of a vertex v in G is denoted by $N_G(v)$ and its degree is $d_G(v) = |N_G(v)|$. For $A \subseteq V$, $N_G(A)$ denotes the set of negihbors outside A of the elements of A. Let $D = (V, E)$ denote a digraph with vertex set V and arc set E. As usual, $N_D^+(v)$ (resp. $N_D^-(v)$) denotes the (first) out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$. $N_D^{++}(v)$ (resp. $N_D^{--}(v)$) denotes the second out-neighborhood (in-neighborhood) of v, which is the set of vertices that are at distance 2 from v (resp. to v). We also denote $d_D^+(v) = |N_D^+(v)|$, $d_D^{++}(v) = |N_D^{++}(v)|$, $d_D^-(v) = |N_D^-(v)|$ and $d_D^{--}(v) = |N_D^{--}(v)|$. We omit the subscript if the digraph (resp. graph) is clear from the context. For short, we write $x \to y$ if the arc $(x, y) \in E$. We say that a vertex v has the *second neighborhood property* (SNP) if $d^+(v) \le d^{++}(v)$.

In 1990, P. Seymour conjectured [2] the following statement:

Conjecture 1. *(The Second Neighborhood Conjecture (SNC))*
Every digraph has a vertex with the SNP.

Seymour's conjecture restricted to tournaments is known as Dean's conjecture [2]. In 1996, Fisher [5] proved Dean's conjecture, thus asserting the SNC for tournaments. Another proof of Dean's conjecture was given by Thomassé and Havet [3], in 2000, using a tool called (local) median order. In 2007, Fidler and Yuster used also median orders to prove SNC for tournaments missing a matching. Ghazal proved the weighted version of SNC for tournaments missing

[1]Department of Mathematics, Faculty of Sciences I, Lebanese University, Hadath, Beirut, Lebanon.
E-mail: salmanghazal@hotmail.com
Institute Camille Jordan, Département de Mathématiques, Université Claude Bernard Lyon 1, France.
E-mail: salman.ghazal@math.univ-lyon1.fr

a generalized star [1].

A median order $L = v_1 v_2 ... v_n$ of a digraph D is an order of the vertices of D the maximizes the size of the set of forward arcs of D, i.e., the set $\{(v_i, v_j) \in E; i < j\}$. In fact, L satisfies the feedback property: For all $1 \leq i \leq j \leq n$:

$$d^+_{[i,j]}(v_i) \geq d^-_{[i,j]}(v_i)$$

and

$$d^-_{[i,j]}(v_j) \geq \omega d^+_{[i,j]}(v_j)$$

where $[i,j] := D[v_i, v_{i+1}, ..., v_j]$.
An order $L = v_1 v_2 ... v_n$ satisfying the feedback property is called a local median order . The last vertex v_n of a weighted local median order $L = v_1 v_2 ... v_n$ of D is called a *feed* vertex of the digraph D [3].

Theorem 1. *[3] Every feed vertex of a tournament has the SNP.*

2 Dependency Digraph

Let $D = (V, E)$ be a digraph. For 2 vertices x and y, we call xy a missing edge if $(x, y) \notin E$ and $(y, x) \notin E$. The missing graph G of D is the graph formed by the missing edges, formally, $E(G)$ is the set of all the missing edge and $V(G)$ is the set of non whole vertices (vertices incident to some missing edges). In this case, we say that D is missing G.

we say that a missing edge $x_1 y_1$ loses to a missing edge $x_2 y_2$ if: $x_1 \rightarrow x_2$, $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \rightarrow y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$. The dependency digraph Δ of D is defined as follows: Its vertex set consists of all the missing edges and $(ab, cd) \in E(\Delta)$ if ab loses to cd. Note that Δ may contain digons.

These digraphs were used in [4] to prove SNC for tournaments missing a matching. However, our defintion is general and is suitable for any digraph.

Definition 1. *[1] A missing edge ab is called* good *if:*
(i) $(\forall v \in V \backslash \{a, b\})[(v \rightarrow a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ or
(ii) $(\forall v \in V \backslash \{a, b\})[(v \rightarrow b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.
If ab satisfies (i) we say that (a, b) is a convenient orientation of ab.
If ab satisfies (ii) we say that (b, a) is a convenient orientation of ab.

The following holds by the definition of good missing edges and losing relation between them.

Lemma 1. *Let D be a digraph and let Δ denote its dependency digraph. A missing edge ab is good if and only if its in-degree in Δ is zero.*

Let $\mathcal{H}$ be a family of digraphs (digons are allowed) and let G be a given graph. We say that G is $\mathcal{H}$-forcing if the dependency digraph of every digraph missing G is a member of $\mathcal{H}$. The set of all $\mathcal{H}$-forcing graphs is denoted by $\mathcal{F}(\mathcal{H})$.

A digraph is trivial if it has no arc.

Proposition 1. *Let $\mathcal{H}$ be a family of digraphs. Then $\mathcal{F}(\mathcal{H})$ is nonempty if and only if $\mathcal{H}$ has a trivial digraph.*

Proof. Let G be a graph and let D be any digraph missing it. Suppose $xy \rightarrow uv$ in Δ, the dependency digraph of D, namely $v \notin N^+(x) \cup N^{++}(x)$. We add to D an extra whole vertex α such that $x \rightarrow \alpha \rightarrow v$. This breaks the arc (xy, uv). Hence, by adding a sufficient number of such vertices, one obtains a digraph whose missing graph is G and such that its dependency digraph is trivial. This establishes the necessary condition.

The converse holds, by absorving that the dependency digraph of any digraph missing a star (edges sharing only one endpoint) is trivial. $\square$

Let $\mathfrak{S}$ denote the class of all trivial digraphs. In [1] Ghazal showed that the only $\mathfrak{S}$-forcing graphs are generalized stars and proved that every digraph missing a generalized star satisfies Seymour's Second Neighborhood Conjecture. In fact, a weighted version the following statement is proved in [1].

Lemma 2. *Let D be a digraph. If all the missing edges of D are good then it has a vertex with the SNP.*

Problem 1. *Let $\vec{\mathcal{P}}$ be the family of all digraphs composed of vertex disjoint directed paths only. Characterize $\mathcal{F}(\vec{\mathcal{P}})$.*

In the next section we present some classes of graphs contained in $\mathcal{F}(\vec{\mathcal{P}})$ and prove that every digraph missing a member of these classes, satisfies SNC.

3   Some Digraphs Missing Graphs of $\mathcal{F}(\vec{\mathcal{P}})$

A comb G is a graph defined as follows:

1) $V(G)$ is disjoint union of three set A, X and Y.

2) $G[X \cup Y]$ is a complete graph.

3) A is stable set with $N(A) = X$ and $N(a) \cap N(b) = \phi$ for any distinct vertices $a, b \in A$.

4) For every $a \in A$, $d(a) = 1$.

Observe that the edges with an end in A form a matching, say M.

Proposition 2. *Combs are $\vec{\mathcal{P}}$-forcing.*

Proof. Let D be a digraph missing a comb G. We follow the previous notations. The only possible arcs of Δ occurs between the edges in M. For $i = 1, 2, 3$ let $a_i x_i \in M$ with $a_i \in A$ and $x_i \in X$. Suppose $a_1 x_1$ loses to the 2 others. Then we have $a_1 \rightarrow x_3$, $x_1 \rightarrow a_2$, $a_2 \notin N^{++}(a_1) \cup N^+(a_1)$ and $x_3 \notin N^{++}(x_1) \cup N^+(x_1)$. Since $a_2 x_3$ is not a missing edge then either $a_2 \rightarrow x_3$ or $a_2 \leftarrow x_3$. Whence, either $x_3 \in N^{++}(x_1) \cup N^+(x_1)$ or $a_2 \in N^{++}(a_1) \cup N^+(a_1)$. A contradiction. Therefore, the maximimum out-degree in Δ is 1. Similarly, the maximum in-degree is 1. Thus Δ is composed of at most vertex disjoint directed paths and directed cycles. Now it is enough to prove that it has no directed cycles. Suppose that $C = a_0 x0, a_1 x_1, ..., a_n x_n$ is a cycle. Then we have $a_{i+1} \notin N^{++}(a_i)$

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and $a_i \leftarrow a_{i+1}$ for all $i < n$. We prove, by induction on i, that $a_i \to a_n$ for all $i < n$. In particular, $a_{n-1} \to a_n$, a contradiction. The case $i = 1$ holds since $a_n x_n$ loses to $a_1 x_1$. Now let $1 < i < n$. By induction hypothesis, $a_{i-1} \to a_n$. Since $a_i \notin N^{++}(a_{i-1})$ and $a_i a_n$ is not a missing edge we must have $(a_i, a_n) \in D$. $\square$

Theorem 2. *Every digraph missing a comb satisfies SNC.*

Proof. Let D be a digraph missing a comb G. We follow the previous notations. Let $P = a_0 x_0, a_1 x_1, ...$ be a maximal directed path in Δ ($a_i \in A$). By lemma 1, $a_0 x_0$ has a convenient orientation. Suppose (a_0, x_0) is a convenient orientation. In this case add (a_{2i}, x_{2i}) and (x_{2i+1}, a_{2i+1}) to D. Otherwise, we orient in the reverse direction. We do this for all such paths of Δ. The obtained digraph D' is missing the complete graph $G[X \cup Y]$. Clearly, all the missing edges of D' are good (in D'), so we give each one a convenient orientation and add it to D'. The obtained digraph T is a tournament. Let L be a local median order of T and let f denote its feed vertex. By theorem 1, f has the SNP in T. We claim that f has the SNP in D as well.

Suppose f is a whole vertex. We show that f gains no vertex in its second out-neighborhood and hence our claim holds. Assume $f \to u \to v \to f$ in T. Since f is whole, $f \to u$ in D. If $u \to v$ in $D' - D$, then it is either a convenient orientation and hence $v \in N^{++}(f)$ or there is a missing edge rs that loses to uv, namely $s \to v$ and $u \notin N^+(s) \cup N^{++}(s)$. However, fs is not a missing edge, then we must have $f \to s$. Whence $v \in N^{++}(f)$. Now, if $u \to v$ in $T - D'$ then $v \in N_{D'}^{++}(f)$. But this case is already discussed. This argument is used implicitly in the rest of the proof.

Suppose $f \in A$. There is a maximal directed path $P = a_0 x_0, ..., a_i x_i, ..., a_k x_k$ with $f = a_i$. If $(x_i, a_i) \in D'$ then $d^+(f) = d_T^+(f) \leq d_T^{++}(f) = d^{++}(f)$. In fact f gains no new first nor second out-neighbor. Otherwise $(a_i, x_i) \in D'$. If $i < k$, f gains only x_i (resp. a_{i+1}) as a first (resp. second) out-neighbor. If $i = k$, we reorient $a_k x_k$ as (x_k, a_k). The same order L is also a local median order of T' the modified tournament. Now f gains no vertex in its second out-neighborhood.

Suppose $f \in X$. There is a maximal directed path $P = a_0 x_0, ..., a_i x_i, ..., a_k x_k$ with $f = x_i$. If $(a_i, x_i) \in D'$ we reorient all the missing edges incident to x_i towards x_i. In this case f gains no new first nor second out-neighbor in the modified tournament. Otherwise $(x_i, a_i) \in D'$. If $i = k$, we reorient all the missing edges incident to x_i towards x_i. In this case f gains no new first nor second out-neighbor in the modified tournament. If $i < k$, we reorient all the missing edges incident to x_i towards x_i except (x_i, a_i). In this case f gains only a_i (resp. x_{i+1}) as a first (resp. second) out-neighbor in the modified tournament.

Suppose $f \in Y$. Reoreient all the missing edges incident to y towards y. In the modified tournament f gains no vertex in its second out-neighborhood.

Therefore D satisfies SNC.

$\square$

A $\tilde{K}^4$ is a graph obtained from the complete graph by removing 2 non adjacent edges. If xy and uv are the removed edges then $\tilde{K}^4$ restricted to $\{x, y, u, v\}$ is a cycle of length 4.

Proposition 3. *The graphs $\tilde{K}^4$ are $\vec{\mathcal{P}}$-forcing.*

Proof. This is clear because the dependency digraph can have at most one arc. $\qquad\square$

Theorem 3. *Every digraph whose missing graph is a $\tilde{K}^4$ satisfies SNC.*

Proof. Let D be a digraph missing a $\tilde{K}^4$. If Δ has no arc then D satisfies SNC by lemma 2. Otherwise, it has exactly one arc, say $xy \to uv$ with $x \to u$ and $v \notin N^{++}(v)$. Note that the cycle $C = xyuv$ is an induced cycle in the missing graph. We may suppose that (x, y) is a convenient orientation. Add (x, y) and (u, v) to D. The rest of the missing edges are good missing edges. So we give them a convenient orientation and add to D. The obtained digraph T is a tournament. Let L be a local median order of T and let f denote its feed vertex. Now f has the SNP in T. We discuss according to f.

Suppose f is a whole vertex. Then f gains no vertex in its second out-neighborhood.

Suppose $f = x$. Reorient all the missing edges incident to x towards x except (x, y). The same order L is a local median order of the modified tournament T'. The only new first (resp. second) out-neighbor of f is y (resp. v).

Suppose $f = y$, u, v or a non whole vertex that does not belong C. Reorient all the missing edges incident to f towards f. In the modified tournament, f gains no vertex in its second out-neighborhood.

$\qquad\square$

A $\tilde{K}^5$ is a graph obtained from the complete graph by removing a cycle of length 5. Note that $\tilde{K}^5$ restricted to the vertices of the removed cycle is also a cycle of length 5.

In the following $ab \to cd$ means ab loses to cd, namely, $a \to c$ and $b \to d$ (the order of the endpoints is considered). Let D be a digraph missing $\tilde{K}^5$ and let Δ denote its dependency digraph. Let $C = xyzuv$ be the induced cyle of length 5 in $\tilde{K}^5$. Checking by cases, we find that Δ has at most 3 arcs. If Δ has exactly 3 arcs then its arcs are (isomorphic to) $uv \to xy \to zu \to vx$ or $uv \to xy \to zu$ and $xv \to zy$.
If Δ has exactly 2 arcs then they are (isomorphic to) $uv \to xy \to zu$ or $uv \to xy$ and $vx \to yz$.
If Δ has exactly 1 arc then it is (isomorphic to) $uv \to xy$. So we have the following.

Proposition 4. *The graphs $\tilde{K}^5$ are $\vec{\mathcal{P}}$-forcing.*

Theorem 4. *Every digraph whose missing graph is a $\tilde{K}^5$ satisfies SNC.*

Proof. Let D be a digraph missing a $\tilde{K}^5$. Let $C = xyzuv$ be the induced cyle of length 5 in $\tilde{K}^5$. If Δ has no arcs then D satisfies SNC by lemma 2.

Suppose Δ has exactly one arc $uv \to xy$. Without loss of generality, we may assume that (u, v) is a convenient orientation. Add (u, v) and (x, y) to D. We give the rest of the missing edges (they are good) convenient orientatins and them add to D. Let L be a local median order of the obtained tournament T and let f denote its feed vertex. f has the SNP in T. If $f = u$ the only new first (resp. second) out-neighbor of u is v (resp. y). Whence f has the SNP in D. Otherwise, we reorient all the missing edges incident to f towards f, if any exist. The same order L is a local median order of the new tournament T' and f has the SNP in T'. However, f gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

Suppose Δ has exactly 2 arcs, say $uv \to xy$ and $vx \to yz$. We may assume that (u, v) is a convenient orientation. Add (u, v) and (x, y) to D. If (v, x) is a convenient orientation, we add (v, x) and (y, z) to D, otherwise we add their reverse. We give the rest of the missing edges (they are good) convenient orientatins and add them to D. Let L be a local median order of the obtained tournament H and let f denote its feed vertex. We reorient every missing edge incident to f, whose other endpoint is not in $\{u, v, x\}$, towards f if any exists . The same L is a local median order of the new tournament T and f has the SNP in T.

If $f \notin \{u, v, x\}$ then it gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = u$, then the only new first (resp. second) out-neighbor of f is v (resp. y), whence f has the SNP in D.

If $f = v$ either $v \to x$ in T and in this case the only new first (resp. second) out-neighbor of v is x (resp. z) or $x \to v$ and in this case f gains neither a new first out-neighbor nor a new second out-neighbor. Whence f has the SNP in D.

If $f = x$ we reorient xy as (y, x). The same L is a local median order of the new tournament T' and f has the SNP in T'. If $v \to x$ in T' then f gains neither a new first out-neighbor nor a new second out-neighbor. Otherwis, $x \to v$ in T' then the only new first (resp. second) out-neighbor of f is v (resp. y). Whence f has the SNP in D.

Suppose Δ has exactly 2 arcs with $uv \to xy \to zu$. We may assume that (u, v) is a convenient orientation. Add (u, v), (x, y) and (z, u) to D. We give the rest of the missing edges (they are good) convenient orientatins and add them to D. Let L be a local median order of the obtained tournament H and let f denote its feed vertex. We reorient every missing edge incident to f, whose other endpoint is not in $\{u, v, x, y, z\}$, towards f if any exists . The same L is a local median order of the new tournament T and f has the SNP in T.

If $f \notin \{u, v, x, y, z\}$ then it gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = u$, then the only new first (resp. second) out-neighbor of f is v (resp. y), whence f has the SNP in D.

If $f = v$ we orient xv as (x, v). The same L is a local median order of the new tournament T' and f has the SNP in T'.

If $f = x$ we orient xv as (v, x). The same L is a local median order of the new tournament T' and f has the SNP in T'. The only new first (resp. second) out-neighbor of f is y (resp. u). Whence f has the SNP in D.

If $f = y$ we orient yz as (z, y). The same L is a local median order of the new tournament T' and f has the SNP in T'. f gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = z$ we orient yz and zu towards z. The same L is a local median order of the new tournament T' and f has the SNP in T'. f gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

Suppose Δ has exactly 3 arcs with $uv \to xy \to zu \to vx$. We may assume that (u, v) is a convenient orientation. Add (u, v), (x, y), (z, u) and (v, x) to D. We give the rest of the missing edges (they are good) convenient orientatins and then add to D. Let L be a local median order of the obtained tournament H and let f denote its feeed vertex. We reorient every missing edge incident to f, whose other endpoint is not in $\{u, v, x, y, z\}$, towards f if any exists . The same L is a local median order of the new tournament T and f has the SNP in T.

If $f \notin \{u, v, x, y, z\}$ then it gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = u$, then the only new first (resp. second) out-neighbor of f is v (resp. y), whence f has the SNP in D.

If $f = v$ we orient xv as (x, v). The same L is a local median order of the new tournament T' and f has the SNP in T'. In this case f gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = x$, the only new first (resp. second) out-neighbor of f is y (resp. u). Whence f has the SNP in D.

If $f = y$ we orient yz as (z, y). The same L is a local median order of the new tournament T' and f has the SNP in T'. f gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = z$ we orient yz towards z. The same L is a local median order of the new tournament T' and f has the SNP in T'. The only new first (resp. second) out-neighbor of f is u (resp. x). Whence f has the SNP in D.

Finally, suppose Δ has exactly 3 arcs with $uv \to xy \to zu$ and $xv \to zy$. We may assume that (u, v) is a convenient orientation. Add (u, v), (x, y) and (z, u) to D. Note that xv is a good missing edge. If (x, v) is a convenient orientation add it with (z, y), otherwise we add the reverse of these arcs. We give the rest of the missing edges (they are good) convenient orientatins and add them to D. Let L be a local median order of the obtained tournament H and let f denote its feeed vertex. We reorient every missing edge incident to f, whose other endpoint is not in $\{u, v, x, y, z\}$, towards f if any exists. The same L is a local median order of the new tournament T and f has the SNP in T.

If $f \notin \{u, v, x, y, z\}$ then it gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

If $f = u$, then the only new first (resp. second) out-neighbor of f is v (resp. y), whence f has the SNP in D.

If $f = v$ either $x \to f = v$ in T and in this case it gains neither a new first out-neighbor nor a new second out-neighbor or $f = v \to x$ and in this case the only new first (resp. second) out-neighbor of f is x (resp. z). Whence f has the SNP in D.

If $f = x$ either $v \to f = x$ in T and in this case the only new first (resp. second) out-neighbor of f is y (resp. z) or $f = x \to v$ and in this case the only new first (resp. second) out-neighbor of f are y and v (resp. z). Whence f has the SNP in D.

If $f = y$ or z, we orient the missing edges incident to f towards f. The same L is a local median order of the new tournament T' and f has the SNP in T'. f gains neither a new first out-neighbor nor a new second out-neighbor. So f has the SNP in D.

$\square$

Digraphs missing a matching are the digraphs with minimum degree $|V(D)| - 2$. These digraphs satisfies SNC [4]. A more general class of digraphs is the class of digraphs with minimum degree at least $|V(D)| - 3$. The missing graph of such a digraph is composed of vertex disjoint directed paths and directed cycles. P_3 is the path of length 3 and C_3, C_4 and C_5 are the cycles of length 3, 4 and 5 respectively. Theorems 2, 3 and 4 implies the following.

Corollary 1. *Every digraph whose missing graph is P_3, C_3, C_4 or a C_5 satisfies SNC.*

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